TORSION OF AN ANISOTROPIC SOLID OF REVOLUTION WITH VARIABLE MODULI OF ELASTICITY

(KRUCHENIE ANIZOTROPNOGO TELA VRASHCHENIIA S PEREMENNYMI MODULIAMI UPRUGOSTI)

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We consider the problem of torsion of a cylindrically anisotropic elastic rod in the shape of a solid of revolution acted upon by forces distributed over its end and side surfaces. It is assumed that at each point of the rod there is a plane of elastic symmetry that passes through its geometric axis, or (a more special case) that the rod is orthotropic, i.e. that there are three planes of elastic symmetry at each point, and that the elasticity moduli in general depend on the cylindrical coordinates r, z. The following cases are studied: that of a conical rod twisted by torques applied at its ends, and of a cylindrical rod acted upon by torsional forces distributed over its side surface.

1. General Equations. We first consider the case of a cylindrically anisotropic elastic rod in the shape of some solid of revolution, whose axis of anisotropy coincides with the geometric z-axis. In the general case, we assume that at each point there is a single plane of elastic symmetry that passes through the z-axis. We also assume that the material of the rod obeys the generalized Hooke's law, and experiences small deformations under stress.

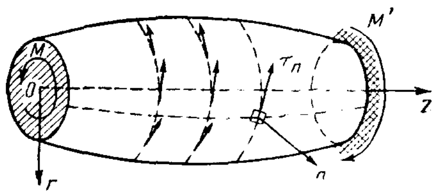

Fig. 1

Let the rod be acted upon by two types of forces: (1) those distributed over the butt ends of the rod and reducible to the torques M and M', and (2) the forces $\tau_n(s)$ distributed over the side surface of the rod, where s is the arc of the axial (meridian) section (Fig.1). As in the case of an isotropic rod, it is possible to construct a complete system of torsion equations, assuming that only two of the six stress components are different from zero, and that only one of the three displacement components is not equal to zero. As our initial assumptions we take (see [1], p.249)

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \quad \tau_{\theta z} = \tau_{\theta z}(r, z), \quad \tau_{r\theta} = \tau_{r\theta}(r, z)$$
$$u_r = w = 0, \quad u_\theta = u_\theta(r, z) \tag{1.1}$$

This eliminates all but three equations of the basic system of elasticity theory

$$\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{\partial (r^2\tau_{\theta z})}{\partial z} = 0 \qquad \frac{\partial}{\partial z}\frac{u_\theta}{r} = \frac{1}{r}(a_{44}\tau_{\theta z} + a_{46}\tau_{r\theta}), \qquad \frac{\partial}{\partial r}\frac{u_\theta}{r} = \frac{1}{r}(a_{46}\tau_{\theta z} + a_{66}\tau_{r\theta}) \tag{1.2}$$

We will consider the deformation coefficients a_{44}, a_{46}, a_{66} to be any differentiable functions of the coordinates r and z; the remaining 10 coefficients a_{ij} from the equations expressing the generalized Hooke's law do not appear in the equations of torsion in any form, and can therefore be either constant or variable. We introduce the stress function $\psi(r,z)$ by means of Equations

$$\tau_{\theta z} = \frac{1}{r^2}\frac{\partial\psi}{\partial r}, \qquad \tau_{r\theta} = -\frac{1}{r^2}\frac{\partial\psi}{\partial z} \tag{1.3}$$

Eliminating the displacement from system (1.2), we obtain the equation for ψ:

$$\frac{\partial}{\partial r}\left[\frac{1}{r^3}\left(a_{44}\frac{\partial\psi}{\partial r} - a_{46}\frac{\partial\psi}{\partial z}\right)\right] - \frac{\partial}{\partial z}\left[\frac{1}{r^3}\left(a_{46}\frac{\partial\psi}{\partial r} - a_{66}\frac{\partial\psi}{\partial z}\right)\right] = 0 \tag{1.4}$$

The boundary conditions on the side surface are of the same form as in the case of an isotropic rod

$$\tau_{\theta z}\cos(n, z) + \tau_{r\theta}\cos(n, r) = \tau_n(s) \quad \text{or} \quad \psi = -r^2\int_0^s \tau_n ds + \psi_0 \tag{1.5}$$

If the torsional forces are distributed over the butt ends only, then along the contour of the meridian section $\psi = \psi_0 = \text{const}$.

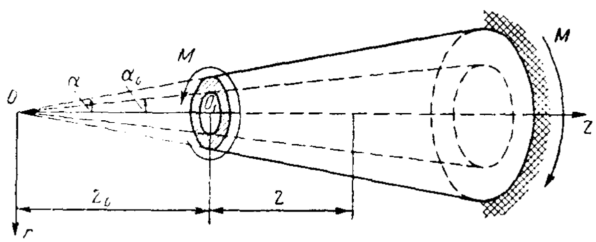

Fig. 2.

Equation (1.4) is simplified in the case of an orthotropic rod. If at each point there are three planes of elastic symmetry, one passing through the axis, another normal to the z-axis, and the third orthogonal to the first two, then $a_{46} = 0$, and in place of a_{44} and a_{66} it is more convenient to introduce shear moduli: $G_1 = 1/a_{44}$ – the shear modulus for planes parallel to the axis of the rod and normal to the direction of r, and $G_2 = 1/a_{66}$ – the shear modulus for the transverse cross-sectional planes, i.e. planes normal to z.

Equation (1.4) then becomes

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3G_1}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^3G_2}\frac{\partial\psi}{\partial z}\right) = 0 \tag{1.6}$$

We shall indicate below some classes of the functions G_1 and G_2, for which the problem of torsion of conical and cylindrical rods may be solved just as simply as for homogeneous and isotropic rods.

2. Torsion of a conical rod. We consider a rod in the form of a truncated circular cone with a fixed base and a free side surface. Over the small end are distributed forces that are reducible to the torque M (Fig. 2). On the side surface

$$\frac{r}{z + z_0} = \tan\alpha \tag{2.1}$$

the stress function ψ takes on a constant value; the difference between the values of this function at the surface and at the axis is proportional to the moment ([1], p.250)

$$\psi(\tan\alpha) - \psi(0) = \frac{M}{2\pi} \tag{2.2}$$

For a rod bounded by two conical surfaces with a common vertex, whose generators from the angles α and α_0 with the z-axis, instead of (2.2) we have

$$\psi(\tan\alpha) - \psi(\tan\alpha_0) = {}^1/_2 M / \pi \tag{2.3}$$

In order for conditions (2.2) or (2.3) to be satisfied, the stress function must be of the form

$$\psi = f(t) \qquad (t = r/\zeta,\ \zeta = z + z_0) \tag{2.4}$$

We point out two cases of shear modulus variation for which the solution can be found in an elementary fashion.

Case 1. The moduli $G_1 = G_1(t)$, $G_2 = G_2(t)$ are arbitrary functions of the ratio t.

Substituting (2.4) into (1.6), we obtain Equation

$$\left(\frac{1}{G_1 t^3} + \frac{1}{G_2 t}\right) f''(t) + \left[\left(\frac{1}{G_1 t^3} + \frac{1}{G_2 t}\right) + \frac{3}{G_2 t^2}\right] f'(t) = 0 \tag{2.5}$$

Integrating, we obtain expressions for $f'(t)$, the stresses, and the displacement

$$f'(t) = A\frac{G_1 G_2 t^3}{G_1 t^2 + G_2}\varphi(t), \qquad \tau_{\theta z} = \frac{A}{\zeta^3}\frac{G_1 G_2 t\varphi(t)}{G_1 t^2 + G_2}, \qquad \tau_{r\theta} = \frac{A}{\zeta^3}\frac{G_1 G_2 t^2 \varphi(t)}{G_1 t^2 + G_2}$$

$$u_\theta = A\int\frac{G_2 t\varphi(t)}{\zeta^3 (G_1 t^2 + G_2)}\,d\zeta + \omega r \qquad \left(\varphi(t) = \exp\left(-3\int\frac{G_1 t}{G_1 t^2 + G_2}\,dt\right)\right) \tag{2.6}$$

The constant A is determined from conditions (2.3) of (2.2)

$$A = \frac{M}{2\pi}\frac{1}{f_1(\tan\alpha) - f_1(\tan\alpha_0)}, \qquad f_1(t) = \frac{G_1 G_2 t^3}{G_1 t^2 + G_2}\varphi(t)\,dt \tag{2.7}$$

With power dependence of the shear moduli on t, when

$$G_1 = g_1 t^n, \qquad G_2 = g_2 t^n \tag{2.8}$$

we obtain the stresses and displacement

$$\tau_{\theta z} = A g_1 g_2 \frac{r^{n+1}}{\zeta^{n-1}(g_1 r^2 + g_2\zeta^2)^{5/2}}, \qquad \tau_{r\theta} = A g_1 g_2 \frac{r^{n+2}}{\zeta^n (g_1 r^2 + g_2 \zeta^2)^{5/2}} \tag{2.9}$$

$$u_\theta = -\frac{Ar}{3(g_1 r^2 + g_2\zeta^2)^{3/2}} + \omega r \tag{2.10}$$

In particular, if the shear moduli are inversely proportional to t^2 ($n=-2$), we obtain

$$\tau_{\theta z} = A g_1 g_2 \frac{\zeta^3}{r(g_1 r^2 + g_2\zeta^2)^{5/2}}, \qquad \tau_{r\theta} = A g_1 g_2 \frac{\zeta^2}{(g_1 r^2 + g_2\zeta^2)^{5/2}} \tag{2.11}$$

$$A = -\frac{3M}{2\pi g_2}\frac{1}{(g_1\tan^2\alpha + g_2)^{3/2} - (g_1\tan^2\alpha_0 + g_2)^{3/2}} \tag{2.12}$$

Case 2. The shear moduli change according to a power law,

$$G_1 = g_1 r^n \zeta^p, \qquad G_2 = g_2 r^n \zeta^p \tag{2.13}$$

In this case there also exists a solution of Equation (1.6) that depends only on the ratio t, and, as may be easily shown, is of the form

$$f'(t) = A t^{n+3}(g_1 t^2 + g_2)^{-N} \qquad (N = {}^1/_2(n + p + 5)) \tag{2.14}$$

The stress components are

$$\tau_{\theta z} = A\frac{r^{n+1}\zeta^{p+1}}{(g_1 r^2 + g_2\zeta^2)^N}, \qquad \tau_{z\theta} = A\frac{r^{n+2}\zeta^p}{(g_1 r^2 + g_2\zeta^2)^N} \tag{2.15}$$

The constant A is found from one of Formulas (2.2) or (2.3), where

$$f_1(t) = \int t^{n+3}(g_1 t^2 + g_2)^{-N}\,dt \tag{2.16}$$

We note that the problem of torsion of solid nad hollow homogeneous isotropic conical rods under the action of various torsional loads is dealt with in great detail in the monograph of Arutiunian and Abramian [2].

3. Torsion of a cylindrical rod due to forces distributed over the side surface. We now consider the case of a cylindrically anisotropic and orthotropic, hollow or solid circular cylindrical rod, one or both of whose ends are fixed, and over whose side sutface are distributed tangential torsional forces that vary along the length of the rod, but remain constant along the cross-sectional circumferences (Fig.3). The shear moduli are specified as functions if a single coordinate r. To be more specific, let the right end $z = l$ be fixed, and the left end $z = 0$ free. We assume that the tangential forces can be represented by Fourier series on the interval $(0, l)$ and expand them in sine series. For the case of the hollow cylinder, we then have the following boundary conditions at the surfaces $r = a$ and $r = b$:

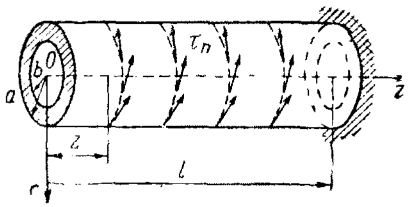

Fig. 3

$$\tau_{r\theta} = \sum_{k=1}^{\infty} m_{ak} \sin\frac{k\pi z}{l} \qquad \text{for } r = a,$$

$$\tau_{r\theta} = \sum_{k=1}^{\infty} m_{bk} \sin\frac{k\pi z}{l} \qquad \text{for } r = b \tag{3.1}$$

In accordance with these conditions, we seek the stress function ψ in the form

$$\psi = R_0(r) + \sum_{k=1}^{\infty} R_k(r)\cos\frac{k\pi z}{l} \tag{3.2}$$

From Equations (1.6) we obtain the expression for R_0

$$R_0 = A_0 \int G_1 r^3 dr + B_0 \tag{3.3}$$

and the equation for R_k

$$R_k'' + G_1 r^3 \left(\frac{1}{G_1 r^3}\right)' R_k' - \left(\frac{k\pi}{l}\right)^2 \frac{G_1}{G_2} R_k = 0 \tag{3.4}$$

With arbitrary G_1 and G_2, this equation is not integrable in general form.

The problem may be solved in an elementary way with the aid of series by representing the shear moduli as power functions of the distance r from the axis.

Let the moduli be given in the form

$$G_1 = g_1 (r / a)^m, \qquad G_2 = g_2 (r / a)^n \tag{3.5}$$

(m, n are arbitrary real numbers; they may be integers, fractions, positive, negative, or zeros). Equation (3.4) becomes

$$R_k'' - \frac{m+3}{r} R_k' - \left(\frac{k\pi}{l}\right)^2 \frac{g_1}{g_2} \frac{r^{m-n}}{a^{m-n}} R_k = 0 \tag{3.6}$$

The integral of this equation is expressed in terms of Bessel functions, and for specific values of m and n — in terms of elementary functions.

We introduce the notation

$$\alpha = -\frac{m+4}{2}, \quad \beta = \frac{m-n+2}{2}, \quad \gamma = \frac{k\pi}{l}\frac{1}{\beta}a^{1/2(n-m)}\left(\frac{g_1}{g_2}\right)^{1/2}, \quad N = \left|\frac{m+4}{m-n+2}\right| \tag{3.7}$$

If N is a fraction, the general solution of Equation (3.6) is expressed in terms of Bessel functions of the imaginary argument $I_N(\gamma r^\beta) = i^{-N} J_N(i\gamma r^\beta)$ and $I_{-N}(\gamma r^\beta)$, and the general expression of the stress function becomes

$$\psi = A_0 r^{m+4} + \sum_{k=1}^{\infty} [A_k I_N(\gamma r^\beta) + B_k I_{-N}(\gamma r^\beta)]\, r^{-\alpha} \cos\frac{k\pi z}{l} \tag{3.8}$$

As in above expression, the constant B_0 will henceforth be omitted, since it has no effect on the stress.

If N is a whole number or zero, the function I_{-N} in Expression (3.8) must be replaced by the MacDonald function $K_N(\gamma r^\beta)$ ([3], pp.46-47). The constants A_k and B_k are determuned from the boundary conditions (3.1) at the cylindrical surfaces.

With the same dependence of the moduli on r, i.e. with $m = n$,

$$\beta = 1, \qquad \gamma = k\pi / l, \qquad N = |\alpha| \tag{3.9}$$

Let us indicate the expressions for ψ for several special cases of power dependence of G_1 and G_2 on r.

1. Linear dependence:

$$G_1 = g_1 r/a, \quad G_2 = g_2 r / a, \quad m = 1, \quad \alpha = -{}^5/_2, \quad N = {}^5/_2 \tag{3.10}$$

Bessel functions whose order is equal to an integer plus one-half are, as we know, expressible in terms of elementary functions ([3], pp.57-70). In the case in question, we have

$$\psi = A_0 r^5 + \sum_{k=1}^{\infty} \{A_k [(3+\gamma^2 r^2)\cosh\gamma r - 3\gamma r \sinh\gamma r] + $$

$$+ B_k [- 3\gamma r\cosh\gamma r + (3+\gamma^2 r^2)\sinh\gamma r]\}\cos\frac{k\pi z}{l} \tag{3.11}$$

2. Inverse proportionality:

$$G_1 = g_1 a / r, \quad G_2 = g_2 a / r, \quad m = -1, \quad \alpha = -{}^3/_2, \quad N = {}^3/_2 \tag{3.12}$$

In this case ψ is also expressed in terms of elementary functions.

$$\psi = A_0 r^3 + \sum_{k=1}^{\infty} [A_k (\sinh\gamma r - \gamma r\cosh\gamma r) + B_k(\cosh\gamma r - \gamma r \sinh\gamma r)]\cos\frac{k\pi z}{l} \tag{3.13}$$

3. Quadratic dependence:

$$G_1 = g_1 (r/a)^2, \quad G_2 = g_2 (r/a)^2, \quad m = 2, \quad \alpha = -3, \quad N = 3 \tag{3.14}$$

$$\psi = A_0 r^6 + \sum_{k=1}^{\infty} [A_k I_3(\gamma r) + B_k K_3(\gamma r)]\, r^3 \cos\frac{k\pi z}{l} \tag{3.15}$$

4. Moduli inversely proportional to the square of the distance:

$$G_1 = g_1 (a/r)^2, \quad G_2 = g_2 (a/r)^2, \quad m = -2, \quad \alpha = -1, \quad N = 1 \tag{3.16}$$

$$\psi = A_0 r^2 + \sum_{k=1}^{\infty} [A_k I_1(\gamma r) + B_k K_1(\gamma r)]\cos\frac{k\pi z}{l} \tag{3.17}$$

Finally, there is the special case where the stress function is expressible in terms of elementary functions.

Let the moduli be given in the form

$$G_1 = g_1 (r/a)^m, \qquad G_2 = g_2 (r/a)^{m+2} \tag{3.18}$$

Equation (3.6) degenerates into an Euler equation, and the general expression for the function ψ is

$$\psi = A_0 r^{m+4} + \sum_{k=1}^{\infty} (A_k r^s + B_k r^{-t}) \cos \frac{k\pi z}{l} \tag{3.19}$$

Here we have introduced the new notation

$$S = \left[\left(\frac{m+4}{2}\right)^2 + \left(\frac{k\pi}{c}\right)^2 g\right]^{1/2} + \frac{m+4}{2}, \qquad t = \left[\left(\frac{m+4}{2}\right)^2 + \left(\frac{k\pi}{c}\right)^2 g\right]^{1/2} - \frac{m+4}{2}, \qquad c = \frac{l}{a}, \qquad g = \left(\frac{g_1}{g_2}\right)^{1/2} \tag{3.20}$$

The above expressions for the stress function permit exact fulfillment of conditions (3.1) at the inner and outer cylindrical surfaces, since for constants A_k, B_k (k = 1, 2, 3, ...) these conditions give us the necessary and sufficient number of equations. At the ends $z = 0$ and $z = l$, the stress is generally reducible to torques. If one end is fixed while the other is free, it is possible to get rid of the "extra" torque by determining the constant A_0 in such a way that the torque at the end is equal to zero (clearly, this is always possible). If both ends are fixed, it is possible to satisfy the required conditions by adding to the solution obtained with the aid of the function ψ the elementary solution for torsion due to the torques acting at the ends.

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